Effective Rate Analysis of MISO Systems over $\alpha-\mu$ Fading Channels

Jiayi Zhang$^{1,2}$, Linglong Dai$^1$, Zhaocheng Wang$^1$
Derrick Wing Kwan Ng$^{2,3}$ and Wolfgang H. Gerstacker$^2$

$^1$Tsinghua National Laboratory for Information Science and Technology (TNList)
Department of Electronic Engineering, Tsinghua University, Beijing 100084, P. R. China

$^2$Institute for Digital Communications, University of Erlangen-Nurnberg, D-91058 Erlangen, Germany

$^3$School of Electrical Engineering and Telecommunications, The University of New South Wales, Australia

San Diego, CA
Dec 07, 2015
Outline

1. Introduction
2. System Model
3. Effective Rate
4. Numerical Results
5. Conclusions
6. References
Effective rate is an appropriate metric to quantify the system performance under QoS limitations and is given by \[ \alpha(\theta) = -\frac{1}{\theta T} \ln \left( \mathbb{E}\{\exp(-\theta TC)\} \right), \quad \theta \neq 0 \] (1)
where \( C \) is the system throughput, \( T \) denotes the block duration and \( \theta \) is the QoS exponent.

For \( \theta \to 0 \), the Effective Rate reduces to the standard ergodic capacity.

**Figure 1: Queuing model**

**Figure 1:** Queuing model

Zhang et al. (Tsinghua & FAU)  
jiayizhang@tsinghua.edu.cn
Introduction II

The $\alpha-\mu$ distribution provides better fit to experimental data than most existing fading models and involves as special cases:

- Rayleigh
- One-sided Gaussian
- Nakagami-$m$
- Weibull
- Exponential
- Gamma

The power PDF of $\alpha-\mu$ variables is given by [2]

$$f_{\gamma_1}(\gamma_1) = \frac{\alpha_1 \gamma_1^{\alpha_1 \mu_1 / 2 - 1}}{2 \beta_1^{\alpha_1 \mu_1 / 2} \Gamma(\mu_1)} \exp \left( - \left( \frac{\gamma_1}{\beta_1} \right)^{\alpha_1 / 2} \right),$$

where $\beta_1 \triangleq \mathbb{E}\{\gamma_1\} \frac{\Gamma(\mu_1)}{\Gamma(\mu_1 + \frac{2}{\alpha_1})}$ with $\mathbb{E}\{\gamma_1\} = \hat{r}_1^2 \frac{\Gamma(\mu_1 + \frac{2}{\alpha_1})}{\left( \mu_1 \frac{2}{\alpha_1} \Gamma(\mu_1) \right)}$, and $\hat{r}_1$ is defined as the $\alpha_1$-root mean value of the envelope random variable $R$, i.e.,

$$\hat{r}_1 = \sqrt[\alpha_1]{\mathbb{E}\left\{ R^{\alpha_1} \right\}}.$$
System Model

We consider a MISO system:

\[ y = hx + n \]  \hspace{1cm} (3)

where \( h \in \mathbb{C}^{1 \times N_t} \) denotes the channel fading vector, \( x \) is the transmit vector with covariance \( \mathbb{E}\{xx^\dagger\} = Q \), and \( n \) represents the AWGN term. The effective rate of the MISO channel can be expressed as [3]

\[ R(\rho, \theta) = -\frac{1}{A} \log_2 \left( \mathbb{E} \left\{ \left( 1 + \frac{\rho}{N_t} hh^\dagger \right)^{-A} \right\} \right) \text{ bits/s/Hz} \]  \hspace{1cm} (4)

where \( A = \frac{\theta TB}{\ln 2} \), with \( B \) denoting the bandwidth of the system, while \( \rho \) is the average transmit SNR.
Lemma 1

[4] The sum of i.i.d. squared $\alpha$-$\mu$ RVs with parameters $\alpha_1$, $\mu_1$, and $\hat{r}_1$, i.e., $\gamma = \sum_{k=1}^{N_t} \gamma_i$, can be approximated by an $\alpha$-$\mu$ RV with parameters $\alpha$, $\mu$ and $\hat{r}$ by solving the following nonlinear equations

\[
\frac{E^2(\gamma)}{E(\gamma^2) - E^2(\gamma)} = \frac{\Gamma^2(\mu + 1/\alpha)}{\Gamma(\mu)\Gamma(\mu + 2/\alpha) - \Gamma^2(\mu + 1/\alpha)},
\]

\[
\frac{E^2(\gamma^2)}{E(\gamma^4) - E^2(\gamma^2)} = \frac{\Gamma^2(\mu + 2/\alpha)}{\Gamma(\mu)\Gamma(\mu + 4/\alpha) - \Gamma^2(\mu + 2/\alpha)},
\]

\[
\hat{r} = \frac{\mu^{1/\alpha}\Gamma(\mu)E(\gamma)}{\Gamma(\mu + 1/\alpha)},
\]

(5)

As such, we can easily obtain the sum PDF as

\[
f_{\gamma}(\gamma) \approx \frac{\alpha \gamma^{\alpha\mu/2 - 1}}{2^{\alpha\mu/2}\Gamma(\mu)} \exp\left(-\left(\frac{\gamma}{\beta}\right)^{\alpha/2}\right).
\]

(6)
Proposition 1

For MISO $\alpha$-$\mu$ fading channels, the effective rate is given by

$$R(\rho, \theta) = \frac{1}{A} - \frac{1}{A} \log_2 \left( \frac{\alpha \sqrt{k} l^{A-1} (N_t \beta / \rho)^{\alpha \mu/2}}{(2\pi)^{l+k/2-3/2} \Gamma(A) \Gamma(\alpha)} \right)$$

$$- \frac{1}{A} \log_2 \left( G_{l,k+1}^{k+1,l} \left[ \frac{(N_t/\rho)^l}{(\beta^{\alpha/2} k)^k} \left| \begin{array}{c} \Delta(l, 1-\alpha \mu/2) \\ \Delta(k, 0), \Delta(l, A-\alpha \mu/2) \end{array} \right. \right) \right), \quad (7)$$

where $G(\cdot)$ is the Meijer’s G-function, $\Delta(\epsilon, \tau) = \frac{\tau}{\epsilon}, \frac{\tau+1}{\epsilon}, \ldots, \frac{\tau+\epsilon-1}{\epsilon}$, with $\tau$ being an arbitrary real value and $\epsilon$ a positive integer. Moreover, $l/k = \alpha/2$, where $l$ and $k$ are both positive integers.

For large values of $l$ and $k$, it is not very efficient to compute (7).
Proposition 2

The effective rate in (7) can be further written in the form of Fox’s $H$-functions by using the Mellin–Barnes integral as

\[
R (\rho, \theta) = \frac{1}{A} \left( 1 - \log_2 \left( \frac{\alpha}{\Gamma (A) \Gamma (\mu)} \right) \right)

- \log_2 \left( \mathcal{H}^{2,1}_{1,2} \left[ \left( \frac{N_t}{\rho \beta} \right)^{\alpha/2} \mid \begin{array}{c} 1, \alpha/2 \\ \mu, 1, (A, \alpha/2) \end{array} \right] \right). \tag{8}
\]

It is worth to mention that (8) is very compact which simplifies the mathematical algebraic manipulations encountered in the effective rate analysis.
Proposition 3

For MISO $\alpha$-$\mu$ fading channels, the effective rate at high SNRs is given by

$$R^\infty (\rho, \theta) \approx \log_2 \left( \frac{\beta \rho}{N_t} \right) - \frac{1}{A} \log_2 \left( \frac{\Gamma (\mu - 2A/\alpha)}{\Gamma (\mu)} \right).$$  \hspace{1cm} (9)

The above result indicates that the high-SNR slope is $S_\infty = 1$, which is independent of $\beta$. The same observations were made in previous works for the Rayleigh, Rician, and Nakagami-$m$ cases.
Proposition 4

For MISO $\alpha$-$\mu$ fading channels, the effective rate at low SNRs is given by

$$ R \left( \frac{E_b}{N_0}, \theta \right) \approx S_0 \log_2 \left( \frac{E_b}{N_0} / \frac{E_b}{N_{0\min}} \right), $$

where

$$ E_b \left/ N_{0\min} \right. = \frac{\Gamma (\mu_1) \ln 2}{\beta_1 \Gamma (\mu_1 + 2/\alpha_1)}, $$

$$ S_0 = \frac{2N_t \Gamma^2 (\mu + 2/\alpha)}{(A + 1) (\Gamma (\mu + 4/\alpha) \Gamma (\mu) - \Gamma^2 (\mu + 2/\alpha)) + N_t \Gamma^2 (\mu + 2/\alpha)}. $$

The minimum $E_b / N_0$ is independent of the delay constraint $A$, whereas the wideband slope $S_0$ is independent of $\beta$, and a decreasing function in $A$, while it is a monotonically increasing function in $N_t$. 
The exact analytical expression is very accurate for all SNRs,

- The high-SNR approximation is quite tight even in moderate SNRs and its accuracy is improved for larger values of the fading parameters,

- An increase of the effective rate is observed as $\alpha$ increases.
An increase of the effective rate is observed as $\alpha$ increases,
Since a large value of $\mu$ results in more multipath components,
A large value of $\alpha$ accounts for a larger fading gain,
The effective rate is a monotonically **decreasing** function of $A$, which implies that tightening the delay constraints reduces the effective rate.

The change of the delay constraint $A$ **does not** affect the minimum $E_b/N_0$, which is $-1.59$ dB in our case.
Conclusions

- **Novel and analytical expressions** of the exact effective rate of MISO systems over i.i.d. $\alpha$-$\mu$ fading channels have been derived by using an $\alpha$-$\mu$ approximation.

- From high-SNR approximation, the effective rate can be improved by utilizing **more transmit antennas** as well as in a propagation environment with **larger values** of $\alpha$ and $\mu$.

- **Our analysis** provides the minimum required transmit energy per information bit for reliably conveying any non-zero rate at **low SNRs**.

- Our analytical results serve as a performance benchmark for our future work on the performance analysis of the **multi-user scenario**.


Thank you!