Low-complexity near-optimal signal detection for uplink large-scale MIMO systems

Xinyu Gao, Linglong Dai, Yongkui Ma and Zhaocheng Wang

The minimum mean square error (MMSE) signal detection algorithm is near-optimal for uplink multi-user large-scale multiple-input–multiple-output (MIMO) systems, but involves matrix inversion with high complexity. It is firstly proved that the MMSE filtering matrix for large-scale MIMO is symmetric positive definite, based on which a low-complexity near-optimal signal detection algorithm by exploiting the Richardson method to avoid the matrix inversion is proposed. The complexity can be reduced from $\mathcal{O}(K^3)$ to $\mathcal{O}(K^2)$, where K is the number of users. The convergence proof of the proposed algorithm is also provided. Simulation results show that the proposed signal detection algorithm converges fast, and achieves the near-optimal performance of the classical MMSE algorithm.

Introduction: Large-scale multiple-input–multiple-output (MIMO) employing hundreds of antennas at the base station (BS) to simultaneously serve multiple users is a promising key technology for fifth generation wireless communications [1]. It can achieve orders of magnitude increases in spectrum and energy efficiency, and one challenging issue to realise such a goal is the low-complexity signal detection algorithm in the uplink, due to the increased dimension of large-scale MIMO systems [2]. The optimal signal detection algorithm is the maximum-likelihood algorithm, but its complexity increases exponentially with the number of transmit antennas, making it impossible for large-scale MIMO systems. The fixed-complexity sphere decoding [3] and tabu search [4] algorithms have been proposed with reduced complexity, but their complexity is unafordable when the dimension of the large-scale MIMO system is large or the modulation order is high [5]. Low-complexity linear detection algorithms such as zero-forcing and minimum mean square error (MMSE) with near-optimal performance have been investigated [2], but these algorithms have to use unfavourable matrix inversion, whose high complexity is still not acceptable for large-scale MIMO systems. Recently, the Neumann series approximation algorithm has been proposed to approximate matrix inversion [6], which converts the matrix inversion into a series of matrix–vector multiplications. However, only marginal complexity reduction can be achieved.

In this Letter, we propose a low-complexity near-optimal signal detection algorithm by exploiting the Richardson method [7] to avoid the complicated matrix inversion. We firstly prove a special property of large-scale MIMO systems that the MMSE filtering matrix is symmetric positive definite, based on which we propose to exploit the Richardson method to avoid the complicated matrix inversion. Then we prove the convergence of the proposed algorithm for any initial solution when the relaxation parameter is appropriate. Finally, we verify through simulations that the proposed signal detection algorithm can efficiently solve the matrix inversion problem in an iterative way until the desired accuracy is attained, and achieve the near-optimal performance of the MMSE algorithm with exact matrix inversion.

Large-scale MIMO system model: We consider an uplink multi-user large-scale MIMO system which employs N antennas at the BS to simultaneously serve K single-antenna users. Usually we have $N > K$, e.g. $N = 128$ and $K = 16$ have been considered [1, 2]. For signal detection, the complex-valued system model can be directly converted to a corresponding real-valued system model, then the estimate of the $2K \times 1$ transmitted signal vector $\hat{\boldsymbol{s}}$ coming from K difference users can be achieved by the classical MMSE algorithm as [2]

$$\hat{\boldsymbol{s}} = (\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} + \sigma^2\boldsymbol{I}_{2K})^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{y} = \boldsymbol{W}^{-1}\hat{\boldsymbol{y}} \quad (1)$$

where $\boldsymbol{H}$ is the $2N \times 2K$ MIMO channel matrix, which can be obtained through frequency-domain and/or time-domain training pilots [8], σ^2 is the additive white Gaussian noise power, $\boldsymbol{I}_{2K}$ is an identity matrix of size $2K \times 2K$, $\boldsymbol{y}$ is the $2N \times 1$ received signal vector at the BS, $\hat{\boldsymbol{y}} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{y}$ can be interpreted as the matched-filter output of $\boldsymbol{y}$ and finally $\boldsymbol{W} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} + \sigma^2\boldsymbol{I}_{2K}$ denotes the MMSE filtering matrix. Note that the direct computation of the matrix inversion $\boldsymbol{W}^{-1}$ requires relatively high complexity of $\mathcal{O}(K^3)$.

Proposed signal detection based on Richardson method: Unlike the conventional (small-scale) MIMO systems with a small number of antennas, large-scale MIMO systems have a special property in that the MMSE filtering matrix $\boldsymbol{W}$ determined by the MIMO channel matrix $\boldsymbol{H}$ is symmetric positive definite, which can be proved as below.

Lemma 1: For signal detection of large-scale MIMO systems, the MMSE filtering matrix $\boldsymbol{W}$ *is symmetric positive definite.*

Proof: The column vectors of the channel matrix $\boldsymbol{H}$ in large-scale MIMO systems are asymptotically orthogonal (i.e. $\mathrm{rank}(\boldsymbol{H}) = 2K$) [2]. Then we have the equation $\boldsymbol{Hr} = 0$ when and only when $\boldsymbol{r}$ is a $2K \times 1$ zero vector. Thus, for an arbitrary nonzero $2K \times 1$ vector $\boldsymbol{r}$, we have

$$(\boldsymbol{Hr})^{\mathrm{H}}\boldsymbol{Hr} = \boldsymbol{r}^{\mathrm{H}}(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H})\boldsymbol{r} > 0 \quad (2)$$

which indicates that the Gram matrix $\boldsymbol{G} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ is positive definite. In addition, as we have $\boldsymbol{G}^{\mathrm{H}} = (\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H})^{\mathrm{H}} = \boldsymbol{G}$, $\boldsymbol{G}$ is also symmetric. Thus, the Gram matrix $\boldsymbol{G}$ is symmetric positive definite. Finally, as the noise power σ^2 is positive, we can conclude that the MMSE filtering matrix $\boldsymbol{W} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} + \sigma^2\boldsymbol{I}_{2K}$ is also a symmetric positive definite matrix. □

The special property that the MMSE filtering matrix $\boldsymbol{W}$ in large-scale MIMO systems is symmetric positive definite inspires us to exploit the Richardson method [7] to efficiently solve (1) in an inversionless way. The Richardson method is used to solve the N-dimension linear equation $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the $N \times N$ symmetric positive definite matrix, $\boldsymbol{x}$ is the $N \times 1$ solution vector and $\boldsymbol{b}$ is the $N \times 1$ measurement vector. The Richardson iteration can be described as

$$\boldsymbol{x}^{(i+1)} = \boldsymbol{x}^{(i)} + w(\boldsymbol{b} - \boldsymbol{Ax}^{(i)}),\ i = 0,\ 1,\ 2,\ \ldots \quad (3)$$

where the superscript i denotes the number of iterations and w represents the relaxation parameter. Since $\boldsymbol{W}$ in (1) is also symmetric positive definite as proved above, we can exploit the Richardson method to estimate the transmitted signal vector $\hat{\boldsymbol{s}}$ without matrix inversion as follows:

$$\boldsymbol{s}^{(i+1)} = \boldsymbol{s}^{(i)} + w(\hat{\boldsymbol{y}} - \boldsymbol{Ws}^{(i)}), \quad i = 0,\ 1,\ 2,\ \ldots \quad (4)$$

where the initial solution $\boldsymbol{s}^{(0)}$ can be usually set as a $2K \times 1$ zero vector without loss of generality as no *a priori* information of the final solution is available [7]. Such an initial solution will not affect the convergence of the Richardson method, since the symmetric positive definite matrix $\boldsymbol{W}$ guarantees the convergence of the Richardson method for any initial solution as we will prove in the following Lemma 2. Consequently, the final accuracy will also not be affected by the initial solution if the number of iterations i is large (e.g. $i = 5$), as will be verified later in the simulation results. Since the relaxation parameter w in (4) plays an important role in convergence, next we prove the convergence of the Richardson method for any initial solution when the relaxation parameter is appropriately selected.

Lemma 2: For the N-dimension linear equation $\boldsymbol{Ax} = \boldsymbol{b}$*, the necessary and sufficient conditions for convergence of the Richardson method is that the relaxation parameter w satisfies* $0 < w < 2/\lambda_1$*, where* λ_1 *is the largest eigenvalue of symmetric positive definite matrix* $\boldsymbol{A}$.

Proof: We define $\boldsymbol{D} = \boldsymbol{I}_N - w\boldsymbol{A}$ and $\boldsymbol{c} = w\boldsymbol{b}$, where $\boldsymbol{D}$ is the iteration matrix. Then the Richardson iteration (3) can be rewritten as

$$\boldsymbol{x}^{(i+1)} = \boldsymbol{Dx}^{(i)} + \boldsymbol{c},\ i = 0,\ 1,\ 2,\ \ldots \quad (5)$$

We call the iteration procedure convergent if $\lim_{i\to\infty} \boldsymbol{s}^{(i)} = \hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{s}} = \boldsymbol{B}\hat{\boldsymbol{s}} + \boldsymbol{c}$ for any initial solution $\boldsymbol{s}^{(0)}$.
The spectral radius of iteration matrix $\boldsymbol{D} \in \mathbb{R}^{N\times N}$ is the non-negative number $\rho(\boldsymbol{D}) = \max_{1\le n\le N}|\mu_n(\boldsymbol{D})|$, where $\mu_n(\boldsymbol{D})$ denotes the nth eigenvalue of $\boldsymbol{D}$, and the necessary and sufficient conditions for the convergence of (5) is that the spectral radius should satisfy [7, Theorem 7.2.2]

$$\rho(\boldsymbol{D}) = \max_{1\le n\le N}|\mu_n(\boldsymbol{D})| < 1 \quad (6)$$

Without loss of generality, we use $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N > 0$ to denote the N eigenvalues of symmetric positive definite matrix $\boldsymbol{A}$, where λ_1 is the largest one. Since $\boldsymbol{D} = \boldsymbol{I} - w\boldsymbol{A}$, we have $\mu_n(\boldsymbol{D}) = 1 - w\lambda_n$, where λ_n is the nth eigenvalue of $\boldsymbol{A}$, which can be substituted into (6), and then we have $0 < w < 2/\lambda_1$. □

Computational complexity: The complexity in terms of the required number of multiplications is analysed for comparison. It can be found from (4) that the ith iteration of the proposed signal detection algorithm involves one multiplication of a $2K \times 2K$ matrix $\boldsymbol{W}$ with a $2K \times 1$ vector $\boldsymbol{s}^{(i)}$, and one multiplication of a constant relaxation parameter w with a $2K \times 1$ vector $\hat{\boldsymbol{y}} - \boldsymbol{W}\boldsymbol{s}^{(i)}$, thus the required number of multiplications is $4K^2 + 2K$ for each iteration.

Table 1 compares the complexity of the conventional Neumann series approximation algorithm [6] and the proposed algorithm based on the Richardson method. It is well known that the complexity of the classical MMSE algorithm is $\mathcal{O}(K^3)$, and Table 1 shows that the conventional Neumann series approximation algorithm can reduce the complexity from $\mathcal{O}(K^3)$ to $\mathcal{O}(K^2)$ when the number of iterations is $i = 2$. However, the complexity is $\mathcal{O}(K^3)$ when $i \geq 3$. Since usually a large value of i is required to ensure the final approximation performance (e.g. $i = 5$ as will be verified later by simulation results), the overall complexity is still $\mathcal{O}(K^3)$, which indicates that only marginal complexity reduction can be achieved. On the contrary, the complexity of the proposed algorithm is reduced from $\mathcal{O}(K^3)$ to $\mathcal{O}(K^2)$ for any arbitrary number of iterations.

Table 1: Computational complexity

	Conventional Neumann series approximation [6]	Proposed algorithm based on Richardson method
$i = 2$	$12K^2 - 4K$	$8K^2 + 4K$
$i = 3$	$8K^3 + 4K^2 - 2K$	$12K^2 + 6K$
$i = 4$	$16K^3 - 4K^2$	$16K^2 + 8K$
$i = 5$	$24K^3 - 12K^2 + 2K$	$20K^2 + 10K$

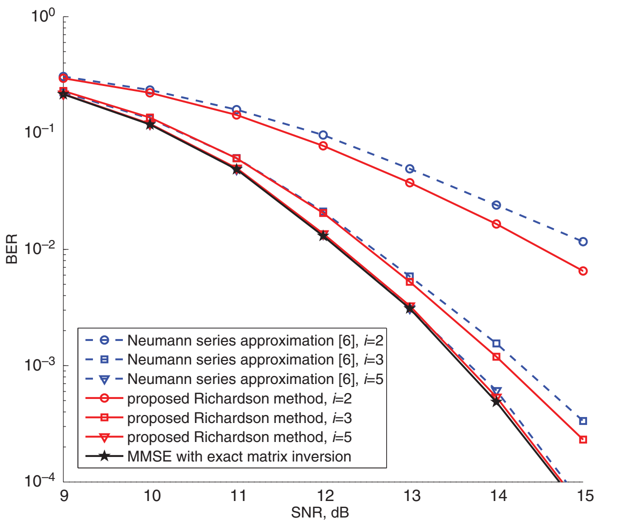

Fig. 1 *BER performance comparison in N × K = 128 × 16 large-scale MIMO system*

Simulation results: The simulation results of the bit error rate (BER) performance against the signal-to-noise ratio (SNR) are provided to compare the proposed signal detection algorithm with the recently proposed Neumann series approximation algorithm [6], whereby the SNR is defined at the receiver. The BER performance of the classical MMSE algorithm with complicated but exact matrix inversion is also included as the benchmark for comparison. We consider an $N \times K = 128 \times 16$ large-scale MIMO system employing the modulation scheme of 64 QAM, and the rate-1/2 convolutional code with $[133_o 171_o]$ polynomial together with a random interleaver. We adopt the flat Rayleigh fading channel. At the receiver, the log-likelihood ratios are extracted from the detected signal for soft-input Viterbi decoding [9]. Through intensive simulations, we find out that when N and K are fixed, the largest eigenvalue of the MMSE filtering matrix $\boldsymbol{W}$ is around a certain value, and accordingly the relaxation parameter is set as $w = 0.00645$ to guarantee convergence.

Fig. 1 shows the BER performance comparison results, where i denotes the number of iterations. It is clear that the BER performance of both the algorithms improves with the number of iterations, but the proposed algorithm outperforms the conventional one when the same number of iterations are used, which indicates that a faster convergence rate can be achieved by the proposed signal detection algorithm. More importantly, when the number of iterations is moderately large (e.g. $i = 5$ in Fig. 1), the proposed algorithm without the complicated matrix inversion can achieve the near-optimal BER performance of the MMSE algorithm with exact matrix inversion.

Conclusions: By fully exploiting the special property that the MMSE filtering matrix in large-scale MIMO systems is symmetric positive definite, we propose a low-complexity near-optimal signal detection algorithm based on the Richardson method to avoid the complicated matrix inversion, which can reduce the complexity from $\mathcal{O}(K^3)$ to $\mathcal{O}(K^2)$. We also prove the convergence of the proposed algorithm for any initial solution when the relaxation parameter is appropriate. Simulation results verify that the proposed algorithm outperforms the conventional method, and achieves the near-optimal performance of the classical MMSE algorithm.

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One or more of the Figures in this Letter are available in colour online.

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